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INNER AMENABLE LOCALLY COMPACT GROUPS

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ABSTRACT. In this paper we study the relationship between amenability, inner amenability and property P of a von Neumann algebra. We give necessary conditions on a locally compact group G to have an inner invariant mean m such that $m(V) = 0$ for some compact neighborhood V of G invariant under the inner automorphisms. We also give a sufficient condition on G (satisfied by the free group on two generators or an I.C.C. discrete group with Kazhdan's property T, e.g., $\mathrm{SL}(n, \mathbb{Z})$, $n \geq 3$) such that each linear form on $L^2(G)$ which is invariant under the inner automorphisms is continuous. A characterization of inner amenability in terms of a fixed point property for left Banach G-modules is also obtained.

INTRODUCTION

Let G be a locally compact group. Then G is called *inner amenable* if there exists a state m on $L^\infty(G)$, such that $m(\pi(a)f) = m(f)$ for all $a \in G$ and $f \in L^\infty(G)$, where

$$\pi(a)f(x) = f(a^{-1}xa), \qquad x \in G.$$

Amenable locally compact groups and *[IN]-groups* are inner amenable. The group G is [IN] if there exists a compact neighborhood V of the identity e in G such that $a^{-1}Va = V$ for all $a \in G$. Furthermore when G is connected, then G is amenable if and only if G is inner amenable (see [17]). A recent account of amenability is given in [21].

Let $\mathscr{M}$ be a von Neumann algebra on a Hilbert space H and let $\mathscr{M}'$ be the commutant of $\mathscr{M}$. For $T \in \mathscr{B}(H)$ (the space of bounded linear operators on H), let C_T be the weak*-closed convex subset of $\mathscr{B}(H)$ generated by $\{U^*TU;\, U \in \mathscr{M}^u\}$, where $\mathscr{M}^u$ is the group of unitary elements in $\mathscr{M}$. (Note that $\mathscr{B}(H)$ has a unique predual [28, p. 47].) $\mathscr{M}$ is said to have *property P* if $C_T \cap \mathscr{M}' \neq \varnothing$ for each $T \in \mathscr{B}(H)$.

Let $\mathrm{VN}(G)$ denote the von Neumann algebra on $L^2(G)$ generated by $\{l_x;\, x \in G\}$ where $l_x h(t) = h(xt)$, $t \in G$. A well-known result of Schwartz [29] asserts

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that if G is discrete, then G is amenable if and only if $\mathrm{VN}(G)$ has property P. In §3 we study the relation between amenability, inner amenability, and property P of a von Neumann algebra determined by G and its action on a locally compact Hausdorff space X. In particular, we provide the missing link in the following well-known implications for a locally compact group G:

$$\begin{tikzcd} & \text{Amenability} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\ \text{Inner Amenability} & & \text{Property } P \text{ (for } \mathrm{VN}(G)) \end{tikzcd}$$

In [20] Paschke proved that if G is an infinite discrete group, then there exists an inner invariant mean on $l^\infty(G)$ different from the point evaluation at the identity if and only if the C^*-algebra generated by the unitaries on $l^2(G)$ corresponding to conjugation by elements in G does not contain the projection on the space $\mathbb{C}\delta_e$, where e is the identity of G. In §4, we find necessary conditions for there to exist an inner invariant mean m on $L^\infty(G)$ such that $m(1_V) = 0$ (when V is a compact neighborhood of G invariant under inner automorphisms). We also give a sufficient condition on G (Theorem 4.4) such that each linear form I on $L^2(G)$ which is invariant under inner automorphisms is continuous and has the form $I(f) = \frac{\alpha}{\lambda(V)} \int_V f\,dx$, where $\alpha = I(1_V)$. In particular (Corollary 4.5 and 4.6) if G is the free group on two generators or a discrete group with Kazhdan's property T and every nontrivial conjugacy class in G is infinite (e.g., $\mathrm{SL}(n, \mathbb{Z})$, $n \geq 3$), then every inner invariant linear form on $l^2(G)$ is continuous. (See [18] for a discussion of similar problems.)

It is well known (see [6 or 26]) that amenability of a locally compact group G may be characterized in terms of fixed points for affine maps on compact convex sets. In §5, we characterize inner amenability of G in terms of a fixed point property for left Banach G-modules. Finally in §6, a few miscellaneous results on inner amenability are stated and proved.

The literature on inner amenability has grown substantially in recent years: see [1, 2, 7, 14, 16, 17, 20, 31].

2. Preliminaries and some notations

Throughout this paper G denotes a locally compact group with a fixed left Haar measure λ. The spaces $L^p(G)$, $1 \leq p \leq \infty$, of measurable functions will be as defined in [13]. For each $a \in G$, $1 \leq p < \infty$, let $\pi(a)$ be the operator on $L^p(G)$ defined by

$$\pi(a)f(t) = f(a^{-1}ta)\Delta^{1/p}(a), \qquad a, t \in G, \ f \in L^p(G),$$

where Δ is the modular function on G. The group G is called *amenable* if there exists a mean m on $L^\infty(G)$ (i.e., $m \in L^\infty(G)$, $m \geq 0$, and $\|m\| = 1$) such that $m(l_a f) = m(f)$ for all $a \in G$ and $f \in L^\infty(G)$. As is well known [8, Theorem 2.2.1], this is equivalent to the existence of a left invariant mean on $U_r(G)$, the space of bounded right uniformly complex-valued continuous functions on G (as defined in [13, p. 21]). All abelian groups and all compact groups are amenable. However, if G contains the free group on two generators as a closed subgroup (e.g., if $G = \mathrm{SL}(2, \mathbb{R})$), then G is not amenable (see [8, 21, 23] for details).

Let $\mathscr{M}$ be a von Neumann algebra on a Hilbert space H. If $\mathscr{M}$ has property P, then there exists a projection of norm one E from $\mathscr{B}(H)$ onto $\mathscr{M}'$ with $E(1) = 1$ (see [28, p. 207, or 24, p. 136]). Von Neumann algebras with this latter property are called injective. As is well known [4], injectivity and property P are equivalent. For a discussion of the various forms of amenability for von Neumann algebras, see [21, 2.35].

If X is a subset of a locally convex space E with topology τ, then $\overline{\mathrm{co}}^{\tau} X$ will denote the closed convex hull of X in E.

3. Inner amenability, amenability, and injectivity

A reference for the definitions below in the discrete cases is Zimmer [32]. Let X be a locally compact Hausdorff space. Let G act invertibly on X on the right such that the mapping $X \times G \to X$ defined by $(x, g) \to x \cdot g$, $x \in X$, $g \in G$, is jointly continuous. Let μ be a nonnegative quasi-invariant Radon measure on X. We define $L^p(X \times G, \mu \times \lambda)$ or simply $L^p(X \times G)$, $1 \leq p \leq \infty$, as the usual L^p-spaces of Borel functions identified when they coincide off a locally $(\mu \times \lambda)$-null set in $X \times G$. For each $a \in G$, define $\mu_a(E) = \mu(Ea)$. Then, by quasi-invariance of μ, we have $\mu_a \ll \mu$ for each $a \in G$ and there is, by the Radon-Nikodým theorem, a locally μ-integrable function $r(\cdot, a)$ such that

$$\int f(xa^{-1})\, d\mu(x) = \int f(x) r(x, a)\, d\mu(x)$$

for all $f \in L^1(X)$ $(= L^1(X, \mu))$. It follows that $r(x, ab) = r(x, a) r(xa, b)$ for $a, b \in G$, and $r(x, e) = 1$. For $u \in G$ and $\phi \in L^\infty(X)$, define the operators U_a, V_a, M_ϕ, and N_ϕ on $L^2(X \times G)$ by

$$\begin{aligned}
U_a f(x, b) &= f(xa, ba) r(x, a)^{1/2} \Delta(a)^{1/2}, \\
V_a f(x, b) &= f(x, a^{-1} b), \\
M_\phi f(x, b) &= \phi(x) f(x, b), \\
N_\phi f(x, b) &= \phi(xb^{-1}) f(x, b),
\end{aligned}$$

where $f \in L^2(X \times G)$.

Then each U_a, V_a is a unitary operator on $L^2(X \times G)$. Let $\mathscr{L}$ be the von Neumann algebra generated by the operators V_a, N_ϕ $(a \in G, \phi \in L_\infty(X))$, and $\mathscr{R}$ be the von Neumann algebra generated by the operators U_a, M_ϕ $(a \in G, \phi \in L_\infty(X))$. If $J \in \mathscr{B}(L^2(X \times G))$ is given by

$$(Jf)(x, b) = f(xb^{-1}, b^{-1}) r(x, b^{-1})^{1/2} \Delta(b^{-1})^{1/2},$$

then $J^2f = f$, $JV_aJ = U_a$, and $JN_\phi J = M_\phi$. So J implements a spatial isomorphism between $\mathscr{L}$ and $\mathscr{R}$. Therefore $\mathscr{L}$ has property P if and only if $\mathscr{R}$ has property P.

If $a \in G$, let δ_a denote the Dirac measure on G concentrated at a. For any $f \in L^\infty(X)$, the function $(\delta_a \square f)(x) = f(xa)$ is defined μ-locally almost everywhere on X (see [10, Lemma 2.1]). Furthermore, $\delta_a \square f \in L^\infty(X)$. A linear functional m on $L^\infty(X)$ is called a *mean* if $m(1) = 1$ and $m(f) \geq 0$ whenever $f \geq 0$. A mean m is G-invariant if $m(\delta_g \square f) = m(f)$ for all $g \in G$.

Also for any $\phi \in L^1(X)$ and $x \in G$, let $\delta_x * \phi \in L^1(X)$ be defined by

$$\delta_x * \phi(\xi) = \left(\frac{d\mu_x}{d\mu}\right)(\xi)\phi(x^{-1}\xi) \quad \mu\text{-a.e. on } X$$

(see [10, Lemma 2.2]), where $\mu_x(E) = \mu(x^{-1}E)$.

Theorem 3.1 below, in the special case where X is a singleton, is proved in [21, p. 85].

Theorem 3.1. *Let G, X, and $\mathscr{L}$ be as above. Then the following are equivalent:*

(a) *G is amenable.*

(b) *$\mathscr{L}$ is injective, $L^\infty(X)$ has a G-invariant mean, and G is inner amenable.*

Proof. (a) $\Rightarrow$ (b) If G is amenable, then G is inner amenable since every invariant mean on G is inner invariant. It follows from [10, Theorem 3.1] that $L^\infty(X)$ has a G-invariant mean.

To see that $\mathscr{L}$ is injective, we first note that the von Neumann algebra generated by $\{N_\phi;\, \phi \in L^\infty(X)\}$ has property P (by the Markov-Katutani fixed point theorem). Hence $\mathscr{D} = \{N_\phi;\, \phi \in L^\infty(X)\}'$ is injective [28, Proposition 4.4.15]. (In fact, $\mathscr{D}$ is the von Neumann algebra generated by the N_ϕ.) It suffices to show that there is a norm one projection from $\mathscr{D}$ onto $\mathscr{L}' = \{V_a;\, a \in G\}' \cap \mathscr{D}$. For then $\mathscr{L}'$ is injective and so $\mathscr{L}$ is also injective.

Let $T \in \mathscr{D}$ and $a \in G$. Then $V_{a^{-1}}TV_a \in \mathscr{D}$. Indeed $\mathscr{D}$ is generated by the N_ϕ's; hence we may assume $T = N_\phi$. If $x \in X$, $b \in G$, and $f \in L^2(X \times G)$, we have

$$\begin{aligned}(V_{a^{-1}}N_\phi V_a f)(x, b) &= (N_\phi V_a f)(x, ab) \\ &= \phi(xb^{-1}a^{-1})(V_a f)(x, ab) = (N_{a^{-1}\phi} f)(x, b),\end{aligned}$$

i.e., $V_{a^{-1}}N_\phi V_a = N_{a^{-1}\phi} \in D$. The result follows.

Let K_T denote the $\overline{\mathrm{co}}^{w^*}\{V_{a^{-1}}TV_a : a \in G\}$ ($w^* = \text{weak}^*$). Then K_T is a w^*-compact convex subset of $\mathscr{D}$. Consider the action of G on K_T defined by

$$(a, S) \to V_{a^{-1}}SV_a.$$

Then the action is separately continuous in the weak operator topology WOT, which agrees with the w^*-topology on K_T. Indeed, if $a_\alpha \to a_0$ and $S \in K_T$, then $V_{a_\alpha} \to V_a$ and $V_{a_\alpha^{-1}} \to V_{a^{-1}}$ in the strong operator topology (SOT). In particular, $SV_{a_\alpha} \to SV_a$ in the SOT, and so $V_{a_\alpha^{-1}}SV_{a_\alpha} \to V_{a^{-1}}SV_a$ in the SOT (since multiplication is jointly continuous on bounded sets in the SOT). Hence $V_{a_\alpha^{-1}}SV_{a_\alpha} \to V_{a^{-1}}SV_a$ in the WOT. Now if $a \in G$, and $S_\alpha \to S$ in the WOT, then for any $\eta, \xi \in L_2(G \times X)$,

$$\langle V_{a^{-1}}S_\alpha V_a\xi, n\rangle = \langle S_\alpha V_a\xi, V_a\eta\rangle \to \langle SV_a\xi, V_a\eta\rangle = \langle V_{a^{-1}}SV_a\xi, \eta\rangle,$$

i.e., $V_{a^{-1}}S_\alpha V_a \to V_{a^{-1}}SV_a$ in the WOT. Apply now Rickert's generalization of Day's fixed point theorem to obtain $S \in K_T$ such that $V_{a^{-1}}SV_a = S$ for all $a \in S$, i.e., $SV_a = V_a$ for all $a \in S$. So $S \in \{V_a : a \in S\}' \cap \mathscr{D}$. Consequently, there exists a projection $Q : \mathscr{D} \to \{V_a : a \in G\}' \cap \mathscr{D}$ such that $Q(T) \in K_T$ for all $T \in \mathscr{D}$, $Q(I) = I$, and $\|Q\| = 1$ by Yeadon's Theorem [30].

(b) $\Rightarrow$ (a) Define a left and a right action of G on $L^\infty(X \times G)$ by

$$(1) \qquad (Fa)(x, b) = F(x, ab), \qquad (aF)(x, b) = F(xa, ba).$$

Using (1) and the equalities $r(x, ab) = r(x, a)r(xa, b)$ a.e. x, and $r(x, e) = 1$ for all $x \in G$, one shows that

$$(2) \qquad \langle F, V_a f\rangle = \langle Fa, f\rangle, \qquad \langle F, U_a f\rangle = \langle a^{-1}F, f\rangle$$

$(F \in L^\infty(X \times G), f \in L^1(X \times G))$. Here (with a slight abuse of notation),

$$V_a f(x, b) = f(x, a^{-1}b), \qquad U_a f(x, b) = f(xa, ba)r(x, a)\Delta(a)$$
$$(f \in L^1(X \times G),\ x \in X,\ a, b \in G).$$

We now show that there exists a positive linear functional m' with $\|m'\| = 1$ such that

$$(3) \qquad m'(aFa^{-1}) = m'(F)$$

for all $a \in G$ and $F \in L^\infty(X \times F)$.

Since $L^\infty(X)$ has a G-invariant mean, an argument similar to that of Namioka [19] shows that there exists a net $\{\phi_\alpha\}$ in $P_1(X) = \{\phi \in L^1(X) : \phi \geq 0$ and $\|\phi\|_1 = 1\}$ such that $\|\delta_a * \phi_\alpha - \phi_\alpha\| \to 0$ for each $a \in G$. Also since G is inner amenable, there exists a net $\{\mu_\beta\}$ in $P_1(G)$ such that $\|\delta_a * \mu_\beta * \delta_{a^{-1}} - \mu_\beta\|_1 \to 0$ (see [17, Proposition 1]). Let

$$m_{\alpha,\beta}(F) = \int F\, d(\phi_\alpha \times \mu_\beta),$$

where $F \in L^\infty(X \times G)$. Then $\{m_{\alpha,\beta}\}$ is bounded in $L^\infty(X \times C)^*$. Furthermore, if $a \in G$ and $F \in L^\infty(X \times G)$, then

$$\begin{aligned}
&|\langle m_{\alpha,\beta}, aFa^{-1}\rangle - \langle m_{\alpha,\beta}, F\rangle| \\
&\quad = \left|\iint F(xa, a^{-1}ba)\, d\phi_\alpha(x)\, d\mu_\beta(b) - \iint F(x, b)\, d\phi_\alpha(x)\, d\mu_\beta(b)\right| \\
&\quad \le \left|\iint F(xa, a^{-1}ba)\, d\phi_\alpha(x)\, d\mu_\beta(b) - \iint F(x, a^{-1}ba)\, d\phi_\alpha(x)\, d\mu_\beta(b)\right| \\
&\qquad + \left|\iint F(x, a^{-1}ba)\, d\phi_\alpha(x)\, d\mu_\beta(b) - \iint F(x, b)\, d\phi_\alpha(x)\, d\mu_\beta(b)\right| \\
&\quad \le \|\delta_a * \phi_\alpha - \phi_\alpha\|\, \|F\|_\infty + \|\delta_a * \mu_\beta * \delta_{a^{-1}} - \mu_\beta\|\, \|F\|_\infty
\end{aligned}$$

which converges to zero. Hence if m' is any weak*-cluster point of the $\{m_{\alpha,\beta}\}$, then m' satisfies (3).

By (3) and an idea of Namioka [19] there exists a net $\{f_\delta\}$ in $L_1(X \times G)$, $f_\delta \ge 0$, $\|f_\delta\|_1 = 1$ such that $\|(V_{a^{-1}} - U_a)b_\delta\|_1 \to 0$. Let $g_\delta = f_\delta^{1/2}$. Note that $g_\delta \in L_2(X \times G)$, $g_\delta \ge 0$, and $\|g_\delta\|_2 = 1$. Then $(V_a f_\delta)^{1/2} = V_a g_\delta$, $(U_a f_\delta)^{1/2} = U_a g_\delta$, and hence

$$\|(V_{a^{-1}} - U_a)g_\delta\|_2 \to 0 \quad \text{for all } a \in G. \tag{4}$$

For each $F \in L^\infty(X \times G)$, let $L_F \in \mathscr{B}(L_2(X \times G))$ be defined by

$$L_F f(x, b) = F(x, b) f(x, b).$$

Then, as readily checked,

$$V_a L_F V_{a^{-1}} = L_{Fa^{-1}} \tag{5}$$

for each $a \in G$. Let H denote the group of unitary elements in the von Neumann algebra $\mathscr{R}$ with the strong operator topology. Let ψ_δ be a function on H defined by $\psi_\delta(F)(U) = \langle UL_F U^* g_\delta, g_\delta\rangle$ $(U \in H)$. Then $\psi_\delta \in U_r(H)$. Also

$$\begin{aligned}
\psi_\delta(Fa^{-1})(U) &= \langle UL_{Fa^{-1}}U^* g_\delta, g_\delta\rangle \\
&= \langle UV_a L_F V_{a^{-1}} U^* g_\delta, g_\delta\rangle \\
&= \langle UL_F U^*(V_{a^{-1}} g_\delta), V_{a^{-1}} g_\delta\rangle
\end{aligned} \tag{6}$$

using (5) and the fact that each V_a is in the commutant of $\mathscr{R}$. Also

$$\psi_\delta(F)V_{a^{-1}}(U) = \langle UL_F U^*(V_a g_\delta), V_a g_\delta\rangle.$$

So

$$\begin{aligned}
&|[\psi_\delta(Fa^{-1}) - \psi_\delta(F)U_{a^{-1}}](U)| \\
&\quad = |\langle UL_F U^* V_{a^{-1}} g_\delta, V_{a^{-1}} g_\delta\rangle - \langle UL_F U^* V_a g_\delta, V_a g_\delta\rangle| \\
&\quad = |\langle UL_F U^*(V_{a^{-1}} - V_a)g_\delta, V_a g_\delta\rangle \\
&\qquad\qquad + \langle UL_F U^* V_{a^{-1}} g_\delta, (V_{a^{-1}} - V_a)g_\delta\rangle| \\
&\quad \le 2\|F\|\, \|V_{a^{-1}} - V_a\|\, \|g_\delta\|_2.
\end{aligned} \tag{7}$$

Since $\mathscr{L}$ is injective, $\mathscr{L}$ must have property P. So $\mathscr{R}$ also has property P. By a result of de la Harpe [12], there exists a left invariant mean m on $U_r(H)$, the space of bounded right uniformly continuous functions on H. Hence using (4) and (7), we have

$$|m(\psi_\delta(Fa^{-1})) - m(\psi_\delta(F))| \to 0.$$

Let $n_\delta = m \circ \psi_\delta$. Then n_δ is a mean on $L^\infty(X \times G)$. Let n be a weak*-cluster point of $\{n_\delta\}$. Then $n(Fa^{-1}) = n(F)$ for all $F \in L^\infty(X \times G)$ and $a \in G$. Define

$$\tilde{n}(\phi) = n(1 \otimes \phi), \qquad \phi \in L^\infty(G).$$

Then $\tilde{n}$ is a left invariant mean on $L^\infty(G)$. Hence G is amenable. □

A well-known result of Schwartz [29] asserts that if G is discrete then G is amenable if and only if VN(G) has property P. Letting G act trivially on a set consisting of one point, we obtain from Theorem 3.1 the following [21, 2.35]:

Corollary 3.2. *Let G be a locally compact group. The following are equivalent:*

(a) *G is amenable.*
(b) *VN(G) is injective and G is inner amenable.*

Corollary 3.3. *Let G be an $[IN]$-group. Then* VN(G) *is injective if and only if G is amenable.*

Corollary 3.4 (Losert and Rindler [17]). *Let G be a connected locally compact group. Then G is amenable if and only if G is inner amenable.*

Proof. If G is inner amenable, let U be a compact neighborhood of G. Then $G_0 = \bigcup_{n=1}^\infty U^n$ is an open (and hence closed), compactly generated subgroup of G. Since G is connected, $G = G_0$. Let K be a compact normal subgroup such that G/K is separable metrizable (see [13, p. 71]). Clearly G/K is connected and inner amenable (Proposition 6.2). However VN(G/K) is injective [5, p. 112]. So G/K is amenable by Theorem 3.1. Hence G is also amenable. □

4. [IN]-GROUPS AND INNER AMENABILITY

Let G be an [IN]-group. Then there exists a compact neighborhood V of e such that $x^{-1}Vx = V$ for each $x \in G$. In this section we find necessary conditions such that there exists an inner invariant mean m on $L^\infty(G)$ with $m(1_V) = 0$. We first establish the following general lemma.

Lemma 4.1. *Let G be a locally compact group. Let $\{\pi, H\}$ be a continuous unitary representation of G. Let $\eta_0 \in H$, $\eta_0 \neq 0$, and $\pi(x)\eta_0 = \eta_0$ for all $x \in G$. Let $H_0 = \{\eta \in H; \langle \eta, \eta_0 \rangle = 0\}$ and $Q \in \mathscr{B}(H)$ be defined by $Q(\eta) = \langle \eta, \eta_0 \rangle \eta_0 / \|\eta_0\|^2$. The following are equivalent:*

(a) *$Q \notin C^*_\pi(G)$ (the C^*-algebra generated by $\{\pi(x); x \in G\}$).*

(b) *There exists a net* $\theta_\alpha \in H_0$ *such that* $\|\theta_\alpha\| = 1$, *and* $\|\pi(x)\theta_\alpha - \theta_\alpha\| \to 0$ *for each* $x \in G$.

(c) *There exists a state* ω *on* $\mathscr{B}(H)$ *such that* $\omega(\pi(x)) = 1$ *for each* $x \in G$ *and* $\omega(Q) = 0$.

Proof. (a) $\Rightarrow$ (b) We follow an idea contained in the proof of [3, Theorem 1.1]. Suppose (b) fails; then we can find $y_1, \dots, y_M \in G$ and $\varepsilon > 0$, such that for all $\theta \in H_0$, $\|\theta\| = 1$, there exists some i, $1 \le i \le M$, such that $\|\pi(y_i)\theta - \theta)\| \ge \varepsilon$. Let $x_1 = e$, the identity of G, and $x_2 = y_1, \dots, x_{M+1} = y_M$. Let $N = M + 1$ and $A = N^{-1}\sum_{k=1}^{N}\pi(x_k)$. We claim that $\|A\|_{H_0} < 1$. If not, we can find a sequence $\theta_n \in H_0$, $\|\theta_n\| = 1$, such that

$$\|A(\theta_n)\|_2^2 = \langle A(\theta_n), A(\theta_n)\rangle = \frac{1}{N^2}\sum_{i,j}\langle \pi(x_j^{-1}x_i)\theta_n, \theta_n\rangle \to 1.$$

Since $|\langle \pi(x_j^{-1}x_i)\theta_n, \theta_n\rangle| \le 1$ for each i, j, we conclude that

$$\mathrm{Re}\langle \pi(x_j^{-1}x_i)\theta_n, \theta_n\rangle \to 1.$$

But then

$$\|\pi(x_i)\theta_n - \pi(x_j)\theta_n\|_2^2 = 2 - \mathrm{Re}\langle \pi(x_j^{-1}x_i)\theta_n, \theta_n\rangle \to 0$$

as $n \to \infty$. In particular, since $x_1 = e$ and $x_{k+1} = y_k$, $k = 1, \dots, M$, we conclude that

$$\lim_n \|\pi(y_k)\theta_n - \theta_n\|_2 = 0 \quad \text{for each } k,\ 1 \le k \le m.$$

This contradicts the choice of $y, \dots, y_M$. Thus $\|A\|_{H_0} < 1$ as claimed.

Observe now that if $\eta \in H$, then

(1) $Q(\eta) = A^m(Q(\eta))$. Indeed, if $x \in G$, then

$$\begin{aligned}\pi(x)Q(\eta) &= \frac{1}{\|\eta_0\|^2}\langle \eta, \eta_0\rangle \pi(x)(\eta_0)\\ &= \frac{1}{\|\eta_0\|^2}\langle \eta, \eta_0\rangle \eta_0 = Q(\eta)\end{aligned}$$

by the invariance of η_0.

(2) $\eta - Q(\eta) \in H_0$. Indeed,

$$\langle \eta - Q(\eta), \eta_0\rangle = \langle \eta, \eta_0\rangle - \frac{1}{\|\eta_0\|^2}\langle \eta, \eta_0\rangle\langle \eta_0, \eta_0\rangle = 0.$$

Hence we have for m fixed and $\eta \in H$,

$$\begin{aligned}\|(A^m - Q)\eta\|_2 &= \|A^m(\eta - Q\eta)\| \quad \text{(by (1))}\\ &\le \|A^m\|_{H_0}\|\eta - Q\eta\| \quad \text{(by (2))}\\ &\le 2\|A\|_{H_0}^m\|\eta\|.\end{aligned}$$

$$\therefore\ \|A^m - Q\| \le 2\|A\|_{H_0}^m \to 0, \quad \text{i.e., } Q \in C_\pi^*(G).$$

(b) ⇒ (c) Let $\omega_\alpha = \langle T\theta_\alpha, \theta_\alpha\rangle$ and ω be a weak*-cluster point of $\{\omega_\alpha\}$ in $\mathscr{B}(H)$. Then clearly $\omega(\pi(x)) = 1$ for each $x \in G$, and $\omega(Q) = 0$.

(c) ⇒ (a) If $X = \sum_{i=1}^n \lambda_i(x_i)$, then

$$\|X - Q\| \geq |\omega(X) - \omega(Q)| = \left|\sum_{i=1}^n \lambda_i\right|$$

and

$$\begin{aligned}\|X - Q\| &\geq |\langle (X-Q)\theta, \theta\rangle| \quad \left(\text{where } \theta = \frac{\eta_0}{\|\eta_0\|}\right)\\ &= |\langle X\theta, \theta\rangle - \langle Q\theta, \theta\rangle|\\ &= \left|\frac{1}{\|\eta_0\|^2}\left\langle \sum \lambda_i \eta_0, \eta_0\right\rangle - \left\langle \frac{1}{\|\eta_0\|^2}\left\langle \frac{\eta_0}{\|\eta_0\|}, \eta_0\right\rangle \eta_0, \frac{\eta_0}{\|\eta_0\|}\right\rangle\right|\\ &= \left|\sum \lambda_i - \frac{1}{\|\eta_0\|^2}\cdot\frac{1}{\|\eta_0\|^2}\langle \eta_0, \eta_0\rangle\langle \eta_0, \eta_0\rangle\right|\\ &= \left|\sum \lambda_i - 1\right|.\end{aligned}$$

Hence $\|X - Q\| \geq \max\{|\sum x_i|, |1 - \sum \lambda_i|\} \geq \frac{1}{2}$. $\therefore$ $Q \notin C_\pi^*(G)$. □

For each $x \in G$, let $\pi(x)f(t) = f(x^{-1}tx)\Delta(x)^{1/2}$, $t \in G$, $f \in L^2(G)$. Then $\{\pi, L^2(G)\}$ is a continuous unitary representation of G. Let $C_\pi^*(G)$ denote the C^*-algebra generated by $\{\pi(x); x \in G\}$ in $\mathscr{B}(L^2(G))$. A discrete version of the following result is proved in [20].

Theorem 4.2. *Let G be a locally compact group and V be a compact neighborhood of e such that $x^{-1}Vx = V$ for all $x \in G$. Let $L_0^2(V) = \{g \in L^2(G); \int_V g(x)\,dx = 0\}$. Consider the following conditions on G:*

(a) *The operator $Q_V(f) = \frac{1}{\lambda(V)}\int_V f(x)\,dx \cdot 1_V$ is not in $C_\pi^*(G)$.*

(b) *There exists a net $\{h_\alpha\}$ in $L_0^2(V)$ such that $\|h_\alpha\|_2 = 1$ and*
$$\|\pi(x)h_\alpha - h_\alpha\|_2 \to 0 \quad \text{for each } x \in G.$$

(c) *There exists a state ω on $\mathscr{B}(H)$ such that $\omega(\pi(x)) = 1$ for each $x \in G$, and $\omega(Q) = 0$.*

(d) *There exists an inner invariant mean m on $L^\infty(G)$ such that $m(l_V) = 0$.*

Then (a) ⇔ (b) ⇔ (c) ⇐ (d).

Proof. That (a) ⇔ (b) ⇔ (c) follows from Lemma 4.1.

(d) ⇒ (b) Indeed, as in Losert and Rindler, there exists a net $\nu_\alpha \in L^1(G)$, $\nu_\alpha \geq 0$, $\|\nu_\alpha\|_1 = 1$, $\nu_\alpha(V) = 0$, and $\|\pi(x)\nu_\alpha\|_1 \to 0$. Let $h_\alpha = \nu_\alpha^{1/2}$; then $\|\pi(x)h_\alpha - h_\alpha\|_1 \to 0$ for all $x \in G$, $\|h_\alpha\|_2 = 1$. Furthermore,

$$\left|\int_V h_\alpha\,dx\right| = \langle h_\alpha 1_V, 1_V\rangle \leq \left(\int_V h_\alpha^2\,dx\right)^{1/2}\lambda(V)^{1/2} = 0,$$

i.e., $h_\alpha \in L_0^2(V)$. □

Open Problem. Is (d) equivalent to the other conditions in Theorem 4.2? (This is the case when G is discrete and $V = \{e\}$ as shown in [20].)

Lemma 4.3. *Let G, $\{\pi, H\}$, η_0, H_0, and Q be as in Lemma 4.1. If $Q \in C^*_\pi(G)$, then each linear form I on H which in invariant under $\{\pi(x): x \in G\}$ is continuous, and has the form*

$$I(\eta) = \frac{\alpha}{\|\eta_0\|^2}\langle \eta, \eta_0\rangle, \quad \text{where } \alpha = I(\eta_0).$$

Proof. As in the proof of Lemma 4.1, (a) $\Rightarrow$ (b), there exists $x_1, \dots, x_{N+1} \in G$, such that $x_1 = e$, and the operator $A = (N+1)^{-1}\sum_{k=1}^{N+1} \pi(x_k)$ satisfies $\|A\|_{H_0} < 1$. In particular, for each $\theta_0 \in H_0$, the series $\theta = \sum_{n=0}^{\infty} A^n(\theta_0)$ converges in H_0. Also,

$$\begin{aligned}\theta_0 = \theta - A\theta &= \frac{N\theta}{N+1} - \frac{1}{N+1}\sum_{i=2}^{N+1} \pi(x_i)\theta \\ &= \sum_{i=2}^{N+1} (\gamma - \pi(x_i)\gamma) \quad \text{with } \gamma = \frac{\theta}{N+1}.\end{aligned}$$

Let $\eta \in H$; then $\theta_0 = \eta - Q(\eta) \in H_0$. So

$$\eta = \eta - Q(\eta) + Q(\eta) = \sum_{i=1}^{n} (\gamma - \pi(x_i)\gamma) + Q(\eta).$$

So if I is invariant on H, then

$$I(\eta) = I(Q(\eta)) = \frac{1}{\|\eta_0\|^2}\langle \eta, \eta_0\rangle I(\eta_0). \quad \square$$

The following is an analogue of the main result in [27].

Theorem 4.4. *Let G be a locally compact group and V be a neighborhood of e such that $x^{-1}Vx = V$ for all $x \in G$, $0 < \lambda(V) < \infty$. If $Q_V \in C^*_\pi(G)$, then each linear form I on $L^2(G)$ which is invariant under inner automorphism is continuous and has the form*

$$I(f) = \frac{\alpha}{\lambda(V)} \cdot \int_V f\,dx, \quad \text{where } \alpha = I(1_V).$$

Proof. This follows from Lemma 4.3. $\square$

Corollary 4.5. *Let G be an I.C.C. discrete group with Kazhdan's property T. Then every inner invariant linear form on $L^2(G)$ is continuous.*

Proof. In this case δ_e is the only inner invariant mean on $L^\infty(G)$. By Paschke's Theorem [20], $Q_V \in C^*_\pi(G)$ when $V = \{e\}$. Apply Theorem 4.4. $\square$

Corollary 4.6. *Let G be the free group on two generators. Then every inner invariant form on $L^2(G)$ is continuous.*

Proof. By the result of Effros [7], δ_e is the only inner invariant mean on $L^\infty(G)$. Apply now Paschke's Theorem [20] and Theorem 4.4. $\square$

Let V be a measurable subset of a locally compact group G. Let $L^2(V) = \{f \in L_2(G) : f_{|_V} = 0\}$. Then $L^2(V)$ is a closed subspace of $L_2(G)$ and $L^2(G) = L^2(V) \oplus L^2(G \sim V)$. Let P_V be the orthogonal projection of $L^2(V)$.

Proposition 4.7. *Let G be a locally compact group. Let V be a measurable subset of G such that $xVx^{-1} = V$ for all $x \in G$. Suppose there exist inner invariant means m, n such that $m(V) = 0$ and $n(G \sim V) = 0$. Then $\|T - P_A\| \geq \frac{1}{2}$ for each $T \in C_\pi^*(G)$.*

Proof. Using an idea of Namioka [19], we may find nets $\{f_\delta\}$ and $\{g_\alpha\}$ of positive norm one functions in $L^1(G)$ such that $f_\delta(A) = 0$, $g_\alpha(G \sim A) = 0$, $\|\pi(x)f_\delta - f_\delta\|_1 \to 0$, and $\|\pi(x)g_\alpha - g_\alpha\|_1 \to 0$ (here $\pi(x)f(t) = f(x^{-1}tx)\Delta(x)$, $f \in L_1(G)$ $(x, t \in G)$). Let $f'_\delta = f_\delta^{1/2}$ and $g'_\alpha = g_\alpha^{1/2}$. Then f'_δ and g'_α are positive norm one functions in $L^2(G)$, $f'_\delta(A) = 0$, $g'_\alpha(G \sim A) = 0$, $\|\pi(x)f'_\delta - f'_\delta\|_2 \to 0$, and $\|\pi(x)g'_\alpha - g'_\alpha\|_2 \to 0$. Let $x_1, \dots, x_n \in G$, $\alpha_1, \dots, \alpha_n \in \mathbb{C}$, and $T = \sum_{i=1}^n \alpha_i \pi(x_i)$. Then

$$\begin{aligned}\|T - P_A\| &\geq \limsup_\delta \|Tf_\delta - P_A f_\delta\|_2 \\ &= \limsup_\delta \left\| \sum_{i=1}^n \alpha_i \pi(x_i) f_\delta \right\|_2 = \left| \sum_{i=1}^n \alpha_i \right| .\end{aligned}$$

Also

$$\begin{aligned}\|T - P_A\| &\geq \limsup_\alpha \|Tg_\alpha - P_A g_\alpha\|_2 \\ &= \limsup_\alpha \left\| \sum_{i=1}^n \alpha_i (\pi(x_i) g_\alpha - g_\alpha) + \left(\sum_{i=1}^n \alpha_i - 1 \right) g_\alpha \right\| \\ &= \left| \sum_{i=1}^n \alpha_i - 1 \right| .\end{aligned}$$

Hence

$$\|T - P_A\| \geq \max \left\{ \left| \sum_{i=1}^n \alpha_i \right| , \left| \sum_{i=1}^n \alpha_i - 1 \right| \right\} \geq \frac{1}{2} . \quad \square$$

5. A FIXED POINT PROPERTY

Let G be a locally compact group. A *left Banach G-module* X is a Banach space X which is a left G-module such that

(i) $\|a \cdot x\| \leq \|x\|$ for all $x \in X$ and $a \in G$.

(ii) For all $x \in X$, the map $a \to a \cdot x$ is continuous from G into X.

In this case, we define $\langle f \cdot a, x \rangle = \langle f, a \cdot x \rangle$ for each $f \in X^*$, $a \in G$, and $x \in X$.

If $\mu \in M(G)$ and $f \in X^*$, we define

$$\langle f \cdot \mu, x \rangle = \int \langle f, a \cdot x \rangle \, d\mu(a), \qquad x \in X .$$

Then $f \cdot \mu \in X^*$, $f \cdot \mu = f \cdot a$ if $\mu = \delta_a$, and $(f \cdot \mu_1) \cdot \mu_2 = f \cdot (\mu_1 * \mu_2)$ for $\mu_1, \mu_2 \in M(G)$. Finally if $a \in G$, $\mu \in M(G)$, and $m \in X^{**}$, we also define

$$\langle a \cdot m, f\rangle = \langle m, f \cdot \alpha\rangle \quad \text{and} \quad \langle \mu \cdot m, f\rangle = \langle m, f \cdot \mu\rangle$$

for all $f \in X^*$.

By the weak* operator topology (W*OT) on $\mathscr{B}(X^{**})$, we shall mean the weak* topology of $\mathscr{B}(X^{**})$ when it is identified with the dual space $(X^{**} \otimes X^*)^*$ in the obvious way. This topology is determined by the seminorms $\{P_{f,m}; f \in X^*, m \in X^{**}\}$ where $p_{f,m}(T) = |\langle Tm, f\rangle|$. Of course, the unit ball in $\mathscr{B}(X^{**})$ is compact in the $\mathrm{W^*OT}$.

For each $\phi \in L^1(G)$, let $T_\phi \in \mathscr{B}(X^{**})$ be defined by $T_\phi(m) = \phi \cdot m$, $m \in X^{**}$. Let $\mathscr{P}_{X^{**}}$ denote the closure of $\{T_\phi; \phi \geq 0, \|\phi\|_1 = 1\}$ in the $\mathrm{W^*OT}$. Then $\mathscr{P}_{X^{**}}$ with the $\mathrm{W^*OT}$ is compact and convex. Also if $a \in G$, let $T_a \in \mathscr{B}(X^{**})$ be defined by $T_a(m) = a \cdot m$, $m \in X^{**}$. Inner amenability can be characterized by the following "fixed point property".

Theorem 5.1. *Let G be a locally compact group. The following are equivalent:*

(a) *G is inner amenable.*
(b) *Whenever X is a left Banach G-module there exists $T \in \mathscr{P}_{X^{**}}$ such that $T_aT = TT_a$ for all $a \in G$.*

Proof. (a) ⇒ (b) Let $\{\phi_\alpha\}$ be a net in $L^1(G)$, $\phi_\alpha \geq 0$, $\|\phi_\alpha\|_1 = 0$, such that $\|\delta_a * \phi_\alpha - \phi_\alpha * \delta_a\|_1 \to 0$ for each $a \in G$ [17, Proposition 1]. Since $\{T_{\phi_\alpha}\}$ is contained in the unit ball of $\mathscr{B}(X^{**})$ and the unit ball is compact in the $\mathrm{W^*OT}$, we may assume by passing to a subnet if necessary that $T_{\phi_\alpha} \to T$ in the $\mathrm{W^*OT}$, $T \in \mathscr{B}(X^{**})$ and $\|T\| \leq 1$. Now if $a \in G$ and $m \in X^{**}$, then

$$\begin{aligned}\|T_aT_{\phi_\alpha}m - T_{\phi_\alpha}T_am\| &= T_{\delta_a*\phi_\alpha}(m) - T_{\phi_\alpha*\delta_a}(m)\| \\ &\leq \|\delta_a * \phi_\alpha - \phi_\alpha * \delta_a\|_1\|m\| \to 0.\end{aligned}$$

On the other hand, $T_aT_{\phi_\alpha} \to T_aT$ and $T_{\phi_\alpha}T_\alpha \to TT_a$ in the $\mathrm{W^*OT}$. In particular $T_aT = TT_a$.

(b) ⇒ (a) Let $X = L^1(G)$ and consider $L^1(G)$ as a left G-module where $a \cdot h = l_{a^{-1}}h$, $a \in G$, $h \in L^1(G)$. Given $m \in L^\infty(G)^*$, $f \in L^\infty(G)$, define $m_L(f) \in L^\infty(G)$ by

$$\langle m_L(f), \phi\rangle = \left\langle m, \frac{1}{\Delta}\tilde{\phi} * f\right\rangle, \qquad \phi \in L_1(G).$$

Define $\langle \widetilde{T}_n(m), f\rangle = \langle n, m_L(f)\rangle$, $n \in L^\infty(G)^*$, $f \in L^\infty(G)$. Then, as readily checked, $\widetilde{T}_\phi = T_\phi$ for each $\phi \in L^1(G)$. Furthermore, the map $n \to \widetilde{T}_n$ from $L^\infty(G)^*$ into $\mathscr{B}(L^\infty(G)^*)$ is continuous when $L^\infty(G)^*$ has the weak*-topology and $\mathscr{B}(L^\infty(G)^*)$ has the $\mathrm{W^*OT}$. Hence

$$\mathscr{P}_{L^\infty(G)^*} = \{\widetilde{T}_n; n \in L^\infty(G)^*, n \geq 0, \text{ and } \|n\| = 1\}.$$

By assumption, there exists $n \in L^\infty(G)^*$, $n \geq 0$, $\|n\| = 1$, such that

$$T_a \widetilde{T}_n = \widetilde{T}_n T_a \quad \text{for all } a \in G. \tag{1}$$

Next we observe that

$$\langle (T_a m)_L(f), \phi\rangle = \langle m_L(f), \phi * \delta_{a^{-1}}\rangle \tag{2}$$

for each $a \in G$, $m \in L^\infty(G)^*$, and $f \in L^\infty(G)$.

Hence if $\{\psi_\alpha\}$ is a bounded approximate identity of $L^1(G)$ and m is a weak* cluster point of ψ_α, then (by (2))

$$\begin{aligned}\langle T_a(m)_L(f), \phi\rangle = \langle m_L(f), \phi * \delta_a\rangle &= \left\langle m, \frac{1}{\Delta}(\phi * \delta_a)^\sim * f\right\rangle \\ &= \lim_\alpha \left\langle \psi_\alpha, \frac{1}{\Delta}(\phi * \delta_a)^\sim * f\right\rangle = \lim_\alpha \langle \phi * \delta_a * \psi_\alpha, f\rangle \\ &= \langle \phi * \delta_a, f\rangle = \langle r_a f, \phi\rangle\end{aligned}$$

for any $f \in L^\infty(G)$ and $\phi \in L^1(G)$, i.e.,

$$T_a(m)_L(f) = r_a f \tag{3}$$

Also

$$\begin{aligned}\langle T_a \widetilde{T}_n(m), f\rangle &= \langle \widetilde{T}_n(m), l_a f\rangle = \langle n \odot m, l_a f\rangle \\ &= \langle n, m_L(l_a f)\rangle = \langle n, l_a m_L(f)\rangle \\ &= \langle l_a^* n, f\rangle = \langle n, l_a f\rangle.\end{aligned}$$

Combining this with (1) and (3), we obtain that $\langle n, l_a f\rangle = \langle n, r_a f\rangle$ for any $f \in L^\infty(G)$ and $a \in G$, i.e., n is an inner invariant mean. □

6. Miscellaneous results

Proposition 6.1. *Let G be a separable connected group. Then the following are equivalent:*

(a) *G admits a countably additive inner invariant mean.*
(b) *G is an $[IN]$-group.*
(c) *G is an extension of a compact group by a vector group.*

Proof. (a) $\Rightarrow$ (b) Let $B(G) = \{x \in G\colon$ the conjugacy class of x has relatively compact closure$\}$. By [9, Theorem 1.4], there exists a layering of G that terminates with the closed subgroup $B(G)$, i.e., a sequence

$$B(G) = X_0 \subset X_1 \subset \cdots \subset X_m = G$$

such that each X_k is a closed subset of G invariant under the inner automorphisms and every point $x \in X_k \sim X_{k-1}$ has a relative neighborhood in X_k with infinitely many disjoint conjugates. Suppose that m is a countably additive inner invariant mean and suppose that $m(B(G)) = 0$. Then $m(X_k \sim X_{k-1}) > 0$ for some k. By separability, there exists a relatively open set U in $X_k \sim X_{k-1}$

with $m(U) > 0$ and a sequence $\{x_n\}$ with $\{x_n U x_n^{-1}\}$ pairwise disjoint. This contradicts $m(G) = 1$. So $m(B(G)) > 0$, and hence $\lambda(B(G)) > 0$, where λ is the left Haar measure on G. Consequently $B(G)$ is an open $[\mathrm{FC}]^-$-subgroup of G. In particular G in an [IN]-group [15, Corollary 2.2].

That (b) $\Leftrightarrow$ (c) for connected groups is well known [11, Corollary 2.8]. Also (b) $\Rightarrow$ (a) is clear. □

Proposition 6.2. *Let G be a locally compact group and H be a closed normal subgroup of G. If G is inner amenable, then G/H is also inner amenable.*

Proof. Define a map $\phi: L^\infty(G/H) \to L^\infty(G)$ by $\phi(f) = f \circ \theta$, where θ is the quotient map of G onto G/H. Then, as is well known (see [25, pp. 66 and 82]), ρ is a linear isometry from $L^\infty(G/H)$ into the subspace A of $L^\infty(G)$, where

$$A = \{f \in L^\infty(G)\,;\, r_x f = f \text{ for all } x \in G\}.$$

Furthermore $\rho(\pi(\dot{x})f) = (\pi(x)f) \circ \theta$ for each $x \in G$, where $\dot{x} = xH$. Let m be an inner invariant mean on $L^\infty(G)$. Define $m'(f) = m(\rho(t))$, $f \in L^\infty(G/H)$. Then, as is readily checked, m' is an inner invariant mean on $L^\infty(G/H)$. □

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